

What bipartite states are lazy

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Abstract. A bipartite state is called lazy if the entropy rate of one subsystem is vanishing for any coupling to the other subsystem. In this paper, we provide a necessary and sufficient condition for a finite-dimensional bipartite state to be lazy, and prove that a two-mode Gaussian state is lazy if and only if it is a direct product state.

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1. Introduction

Quantum correlation manifests abundant structures and powerful applications [1, 2]. How many kinds of quantum correlation and how to characterize them are therefore quite fundamental questions. Entanglement and discord, as two kinds of quantum correlation, drawing strong attention, have been intensively studied, and still in active research (for examples see [3, 4, 5, 6]).

A bipartite state is called lazy, if the entropy rate of one subsystem is zero for any coupling to the other subsystem. In [7], the authors established necessary and sufficient conditions for a state to be lazy. In [8], the authors showed that almost all states are pretty lazy. It is shown that a maximally entangled pure state is lazy [9], this indicates that the correlation described by lazy states do not coincide the correlation described by entanglement. In [10], by investigating some 2-qubit states, the authors showed that there indeed exist many lazy states which are entangled, and exist many separable states which are not lazy.

This paper consider the more general cases. We explore the conditions of a state to be lazy for arbitrary finite-dimensional bipartite quantum states and two-mode Gaussian states.

This paper is organized as follows. In Section 2, we establish a necessary and sufficient condition for a state to be lazy for arbitrary finite-dimensional bipartite quantum states. In Section 3, we prove that a two-mode Gaussian state is lazy if and only if it is a direct product state. In section 4, we briefly summary this paper.

As preparations, we briefly review the definition of bipartite lazy state and introduce some notations. Suppose that quantum systems A and B are described by the Hilbert

spaces H^A and H^B respectively, the composite system AB is then described by the Hilbert space $H^{AB} = H^A \otimes H^B$. Let $n_A = \dim H^A$, $n_B = \dim H^B$, being finite or infinite. A state ρ^{AB} on H^{AB} is called a lazy state with respect to A if [7]

$$C_A(\rho^{AB}) = [\rho^{AB}, \rho^A \otimes I^B] = 0, \quad (1)$$

where $\rho^A = \text{tr}_B \rho^{AB}$, I^B is the identity operator on H^B . We often omit I^A and I^B without any ambiguity. Note that $[\rho^{AB}, \rho^A \otimes I^B] = 0$ keeps invariant under locally unitary transformations.

An important physical interpretation of lazy states is that the entropy rate of A is zero in the time evolution under any coupling to B [7]

$$C_A(\rho^{AB}(t)) = 0 \Leftrightarrow \frac{d}{dt} \text{tr}_A[\rho^A(t) \log_2 \rho^A(t)] = 0. \quad (2)$$

2. lazy states of finite-dimensional bipartite systems

When $n_A = \dim H^A$, $n_B = \dim H^B$ are finite, any state ρ^{AB} can be expressed as [11]

$$\rho^{AB} = \frac{1}{n_A n_B} (I^A \otimes I^B + \sum_{i=1}^{n_A^2-1} x_i \sigma_i \otimes I^B + \sum_{j=1}^{n_B^2-1} y_j I^A \otimes \tau_j + \sum_{i=1}^{n_A^2-1} \sum_{j=1}^{n_B^2-1} T_{ij} \sigma_i \otimes \tau_j). \quad (3)$$

In Eq.(3), we used the $\{\sigma_i\}_{i=1}^{n_A^2-1}$ ($\{\tau_j\}_{j=1}^{n_B^2-1}$ similarly) defined as

$$\{\sigma_i\}_{i=1}^{n_A^2-1} = \{w_l, u_{jk}, v_{jk}\}, \quad (4)$$

$$w_l = -\sqrt{\frac{2}{l(l+1)}} (P_{11} + P_{22} + \dots + P_{ll} - l P_{l+1, l+1}), \quad 1 \leq l \leq n_A - 1, \quad (5)$$

$$u_{jk} = P_{jk} + P_{kj}, v_{jk} = i(P_{jk} - P_{kj}), \quad 1 \leq j < k \leq n_A, \quad (6)$$

where $P_{jk} = |j\rangle\langle k|$ with $\{|j\rangle\}_{j=1}^{n_A}$ an orthonormal basis for H^A , $\{w_l, u_{jk}, v_{jk}\}$ is arranged for any fixed order. $\{\sigma_i\}_{i=1}^{n_A^2-1}$ are the traceless generators of $su(n_A)$ algebra, and fulfill the relations [11, 12]

$$\text{tr} \sigma_i = 0, \quad \text{tr}(\sigma_i \sigma_j) = 2\delta_{ij}, \quad [\sigma_i, \sigma_j] = 2i \sum_{k=1}^{n_A^2-1} f_{ijk} \sigma_k, \quad (7)$$

where $[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i$, f_{ijk} is totally antisymmetric in the subindices $\{ijk\}$. When $n_A = 2$, $\{\sigma_i\}_{i=1}^3$ are the well known Pauli operators, f_{ijk} the permutation symbol.

Now we derive the condition $[\rho^{AB}, \rho^A] = 0$ for the state ρ^{AB} expressed in the form in Eq.(3). From Eq.(3), we have

$$\rho^A = \frac{1}{n_A} (I^A + \sum_{i=1}^{n_A^2-1} x_i \sigma_i \otimes I^B). \quad (8)$$

Then

$$[\rho^{AB}, \rho^A] = \frac{1}{n_A n_B} \sum_{ik=1}^{n_A^2-1} \sum_{j=1}^{n_B^2-1} [T_{ij} \sigma_i \otimes \tau_j, x_k \sigma_k \otimes I^B] \quad (9)$$

$$= \frac{1}{n_A^2 n_B} \sum_{ik=1}^{n_A^2-1} \sum_{j=1}^{n_B^2-1} T_{ij} x_k [\sigma_i, \sigma_k] \otimes \tau_j \quad (10)$$

$$= \frac{2i}{n_A^2 n_B} \sum_{ikl=1}^{n_A^2-1} \sum_{j=1}^{n_B^2-1} T_{ij} x_k f_{ikl} \sigma_l \otimes \tau_j. \quad (11)$$

Thus $[\rho^{AB}, \rho^A] = 0$ leads to $\sum_{ik=1}^{n_A^2-1} T_{ij} x_k f_{ikl} = 0$ for any l, j .

Proposition 1. ρ^{AB} in the form in Eq.(3) is lazy respect to A if and only if

$$\sum_{ik=1}^{n_A^2-1} T_{ij} x_k f_{ikl} = 0 \text{ for any } 1 \leq l \leq n_A^2 - 1, 1 \leq j \leq n_B^2 - 1. \quad (12)$$

The lazy states respect to B have the similar result.

Example 1. As a demonstration, we consider a special class of 3×3 states

$$\rho^{AB} = \frac{1}{9} (I^A \otimes I^B + \sum_{i=1}^8 x_i \sigma_i \otimes I^B + \sum_{j=1}^8 y_j I^A \otimes \sigma_j + \sum_{k=1}^8 \lambda_k \sigma_k \otimes \sigma_k), \quad (13)$$

where $\lambda_k \neq 0$ for all k .

$n_A = 3$, then [12] $f_{147} = 1, f_{216} = f_{315} = f_{324} = f_{257} = f_{376} = f_{546} = 1/2, f_{368} = f_{258} = \sqrt{3}/2$, notice that f_{ikl} is totally antisymmetric, thus $f_{417} = 1$, etc. Otherwise $f_{ikl} = 0$.

For Eq.(13), for any j, l , Eq.(12) leads to $\sum_{k=1}^8 \lambda_j x_k f_{jkl} = 0$, thus $\sum_{k=1}^8 x_k f_{jkl} = F_{jl} = 0$. We explicitly write out the matrix $F = (F_{jl}) = 0$ as

$$\begin{pmatrix} 0 & \frac{x_6}{2} & \frac{x_5}{2} & -x_7 & -\frac{x_3}{2} & -\frac{x_2}{2} & x_4 & 0 \\ -\frac{x_6}{2} & 0 & \frac{x_4}{2} & -\frac{x_3}{2} & -\frac{x_7}{2} - \frac{\sqrt{3}x_8}{2} & \frac{x_1}{2} & \frac{x_5}{2} & \frac{\sqrt{3}x_5}{2} \\ -\frac{x_5}{2} & -\frac{x_4}{2} & 0 & \frac{x_2}{2} & \frac{x_1}{2} & \frac{x_7}{2} - \frac{\sqrt{3}x_8}{2} & -\frac{x_6}{2} & \frac{\sqrt{3}x_6}{2} \\ x_7 & \frac{x_3}{2} & -\frac{x_2}{2} & 0 & \frac{x_6}{2} & -\frac{x_5}{2} & -x_1 & 0 \\ \frac{x_3}{2} & \frac{x_7}{2} + \frac{\sqrt{3}x_8}{2} & -\frac{x_1}{2} & -\frac{x_6}{2} & 0 & \frac{x_4}{2} & -\frac{x_2}{2} & -\frac{\sqrt{3}x_2}{2} \\ \frac{x_2}{2} & -\frac{x_1}{2} & -\frac{x_7}{2} + \frac{\sqrt{3}x_8}{2} & \frac{x_5}{2} & -\frac{x_4}{2} & 0 & \frac{x_3}{2} & -\frac{\sqrt{3}x_3}{2} \\ -x_4 & -\frac{x_5}{2} & \frac{x_6}{2} & x_1 & \frac{x_2}{2} & -\frac{x_3}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{3}x_5}{2} & -\frac{\sqrt{3}x_6}{2} & 0 & \frac{\sqrt{3}x_2}{2} & \frac{\sqrt{3}x_3}{2} & 0 & 0 \end{pmatrix} = 0. \quad (14)$$

Consequently, ρ^{AB} in Eq.(13) with all $\lambda_k \neq 0$ is lazy if and only if $x_i = 0$ for all i .

3. Two-mode lazy Gaussian states

Gaussian states are of great practical relevance in quantum information processing (for recent reviews see [13, 14, 15] etc). The entanglement and discord of two-mode Gaussian states have been studied [16, 17, 18, 19]. In this section, we explore that what two-mode Gaussian states are lazy.

Consider system A with continuous variables $\{x_1, p_1\}$ and system B with continuous variables $\{x_2, p_2\}$ satisfying $[x_1, p_1] = [x_2, p_2] = i$ and $[x_1, x_2] = [x_1, p_2] = [p_1, x_2] =$

$[p_1, p_2] = 0$. The creation and annihilation operators are defined as $a_j = (x_j + ip_j)/\sqrt{2}$, $a_j^\dagger = (x_j - ip_j)/\sqrt{2}$, $j = 1, 2$. Where $+$ denotes adjoint. The Wigner characteristic function $\chi(\rho^{AB}, \lambda_1, \lambda_2)$ of the two-mode state ρ^{AB} is defined as

$$\chi(\rho^{AB}, \lambda_1, \lambda_2) = \text{tr}(\rho^{AB} D(\lambda_1) D(\lambda_2)), \quad (15)$$

where

$$D(\lambda_j) = \exp(\lambda_j a_j^\dagger - \lambda_j^* a_j) \quad (16)$$

is the displacement operator, λ_j^* is the complex conjugate of λ_j .

ρ^{AB} is Gaussian if $\chi(\rho^{AB}, \lambda_1, \lambda_2)$ has the form

$$\chi(\rho^{AB}, \lambda_1, \lambda_2) = \exp\left[-\frac{1}{2}(\lambda_1^I, \lambda_1^R, \lambda_2^I, \lambda_2^R) V (\lambda_1^I, \lambda_1^R, \lambda_2^I, \lambda_2^R)^t - i(d_1, d_2, d_3, d_4)(\lambda_1^I, \lambda_1^R, \lambda_2^I, \lambda_2^R)^t\right], \quad (17)$$

where the covariance matrix V is a real and symmetric matrix satisfying the uncertainty relation [20], (d_1, d_2, d_3, d_4) is a real vector, $\lambda_j = \lambda_j^R + i\lambda_j^I$.

We now prove Proposition 2 below.

Proposition 2. A two-mode Gaussian state is lazy if and only if it is a direct product state.

Proof. It is known that up to locally unitary transformations, $\chi(\rho^{AB}, \lambda_1, \lambda_2)$ of a Gaussian state ρ^{AB} can be written in the form [16]

$$\chi(\rho^{AB}, \lambda_1, \lambda_2) = \exp\left[-\frac{1}{2}(\lambda_1^I, \lambda_1^R, \lambda_2^I, \lambda_2^R) M (\lambda_1^I, \lambda_1^R, \lambda_2^I, \lambda_2^R)^t\right], \quad (18)$$

$$M = \begin{pmatrix} n & 0 & c & 0 \\ 0 & n & 0 & c' \\ c & 0 & m & 0 \\ 0 & c' & 0 & m \end{pmatrix}, n \geq 1, m \geq 1. \quad (19)$$

Recall the identities (see for example [21])

$$\rho^{AB} = \int \frac{d^2\lambda_1}{\pi} \frac{d^2\lambda_2}{\pi} \chi(\rho^{AB}, \lambda_1, \lambda_2) D(-\lambda_1) D(-\lambda_2), \quad (20)$$

$$\chi(\rho^A, \lambda_1) = \chi(\rho^{AB}, \lambda_1, 0), \quad (21)$$

$$\rho^A = \int \frac{d^2\lambda_1}{\pi} \chi(\rho^A, \lambda_1) D(-\lambda_1), \quad (22)$$

$$D(\lambda_1) D(\mu_1) = \exp\left(\frac{\lambda_1 \mu_1^* - \lambda_1^* \mu_1}{2}\right) D(\lambda_1 + \mu_1). \quad (23)$$

Notice that the integrations in this section are all over $(-\infty, \infty)$.

From Eqs.(20-23), we get

$$\begin{aligned} [\rho^{AB}, \rho^A] &= \int \frac{d^2\lambda_1}{\pi} \frac{d^2\lambda_2}{\pi} \frac{d^2\mu_1}{\pi} \chi(\rho^{AB}, \lambda_1, \lambda_2) \chi(\rho^{AB}, \mu_1) [D(-\lambda_1) D(-\lambda_2), D(-\mu_1)] \\ &= \int \frac{d^2\lambda_1}{\pi} \frac{d^2\lambda_2}{\pi} \frac{d^2\mu_1}{\pi} \chi(\rho^{AB}, \lambda_1, \lambda_2) \chi(\rho^{AB}, \mu_1) [D(-\lambda_1), D(-\mu_1)] D(-\lambda_2) \\ &= \int \frac{d^2\lambda_1}{\pi} \frac{d^2\lambda_2}{\pi} \frac{d^2\mu_1}{\pi} \chi(\rho^{AB}, \lambda_1, \lambda_2) \chi(\rho^{AB}, \mu_1) (e^{\lambda_1 \mu_1^* - \lambda_1^* \mu_1} - e^{-\lambda_1 \mu_1^* + \lambda_1^* \mu_1}) \\ &\quad \cdot D(-\lambda_1 - \mu_1) D(-\lambda_2). \end{aligned} \quad (24)$$

For any two-mode linear operator σ^{AB} , using the Glauber–Sudarshan P function, σ^{AB} can be expressed as [22, 23, 24]

$$\sigma^{AB} = \int d^2\alpha d^2\beta P(\alpha, \beta) |\alpha\beta\rangle \langle \alpha\beta|, \quad (25)$$

where,

$$P(\alpha, \beta) = \frac{1}{\pi^4} e^{|\alpha|^2 + |\beta|^2} \int d^2u d^2v \langle -u, -v | \sigma^{AB} | uv \rangle e^{u^* \alpha - u \alpha^*} e^{v^* \beta - v \beta^*} e^{|u|^2 + |v|^2}, \quad (26)$$

$$\langle -u, -v | \sigma^{AB} | uv \rangle = e^{-|u|^2 - |v|^2} \int d^2\alpha d^2\beta P(\alpha, \beta) e^{\alpha^* u - \alpha u^*} e^{\beta^* v - \beta v^*} e^{-|\alpha|^2 - |\beta|^2}, \quad (27)$$

$|\alpha\rangle, |u\rangle$ are any coherent states of A , $|\beta\rangle, |v\rangle$ are any coherent states of B .

From Eqs.(25-27), we see that

$$\sigma^{AB} = 0 \Leftrightarrow P(\alpha, \beta) = 0 \text{ for any } \alpha, \beta \Leftrightarrow \langle -u, -v | \sigma^{AB} | uv \rangle \text{ for any } u, v. \quad (28)$$

Then

$$[\rho^{AB}, \rho^A] = 0 \Leftrightarrow \langle -u, -v | [\rho^{AB}, \rho^A] | uv \rangle = 0 \text{ for any } u, v. \quad (29)$$

From Eq.(24), $\langle -u, -v | [\rho^{AB}, \rho^A] | uv \rangle = 0$ reads

$$\int \frac{d^2\lambda_1}{\pi} \frac{d^2\lambda_2}{\pi} \frac{d^2\mu_1}{\pi} \chi(\rho^{AB}, \lambda_1, \lambda_2) \chi(\rho^{AB}, \mu_1) (e^{\lambda_1 \mu_1^* - \lambda_1^* \mu_1} - e^{-\lambda_1 \mu_1^* + \lambda_1^* \mu_1}) \cdot \langle -u | D(-\lambda_1 - \mu_1) | u \rangle \langle -v | D(-\lambda_2) | v \rangle = 0. \quad (30)$$

Using the relations (here $\{|i\rangle\}_i$ are the number states)

$$|u\rangle = \exp\left(-\frac{|u|^2}{2}\right) \sum_{i=0}^{\infty} |i\rangle = D(u) |0\rangle, \quad D(u)^+ = D(-u), \quad (31)$$

$$D(u) D(-\lambda_1 - \mu_1) D(u) = D(2u - \lambda_1 - \mu_1), \quad (32)$$

and the counterparts for system B , Eq.(30) becomes

$$\int \frac{d^2\lambda_1}{\pi} \frac{d^2\lambda_2}{\pi} \frac{d^2\mu_1}{\pi} \chi(\rho^{AB}, \lambda_1, \lambda_2) \chi(\rho^{AB}, \mu_1) (e^{\lambda_1 \mu_1^* - \lambda_1^* \mu_1} - e^{-\lambda_1 \mu_1^* + \lambda_1^* \mu_1}) \cdot \exp\left[-\frac{|2u - \lambda_1 - \mu_1|^2}{2} - \frac{|2v - \lambda_2|^2}{2}\right] = 0. \quad (33)$$

Inserting $\lambda_j = \lambda_j^R + i\lambda_j^I$, $\mu_1 = \mu_1^R + i\mu_1^I$, $u = u^R + iu^I$, $v = v^R + iv^I$ into Eq.(33), we get

$$\begin{aligned} & \int \frac{d^2\lambda_1}{\pi} \frac{d^2\lambda_2}{\pi} \frac{d^2\mu_1}{\pi} \exp\left[-\frac{1}{2} \vec{X}^t A_1 \vec{X} + 2 \vec{B}^t \vec{X}\right] \\ &= \int \frac{d^2\lambda_1}{\pi} \frac{d^2\lambda_2}{\pi} \frac{d^2\mu_1}{\pi} \exp\left[-\frac{1}{2} \vec{X}^t A_2 \vec{X} + 2 \vec{B}^t \vec{X}\right], \end{aligned} \quad (34)$$

where

$$\vec{X} = (\lambda_1^I, \lambda_1^R, \lambda_2^I, \lambda_2^R, \mu_1^I, \mu_1^R)^t, \quad \vec{B} = (u^I, u^R, v^I, v^R, u^I, u^R)^t, \quad (35)$$

$$A_1 = \begin{pmatrix} 2n+1 & 0 & c & 0 & 1 & 2i \\ 0 & 2n+1 & 0 & c' & -2i & 1 \\ c & 0 & m+1 & 0 & 0 & 0 \\ 0 & c' & 0 & m+1 & 0 & 0 \\ 1 & -2i & 0 & 0 & 1 & 0 \\ 2i & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (36)$$

$$A_2 = \begin{pmatrix} 2n+1 & 0 & c & 0 & 1 & -2i \\ 0 & 2n+1 & 0 & c' & 2i & 1 \\ c & 0 & m+1 & 0 & 0 & 0 \\ 0 & c' & 0 & m+1 & 0 & 0 \\ 1 & 2i & 0 & 0 & 1 & 0 \\ -2i & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (37)$$

t denotes transpose.

Recall the identity (see for example [25])

$$\begin{aligned} & \int dz_1^R dz_1^I dz_1^R dz_1^I \dots dz_N^R dz_N^I \exp[-\vec{Z}^+ A_3 \vec{Z} + \vec{W}^+ \vec{Z} + \vec{Z}^+ \vec{W}] \\ &= \pi^N \det(A_3^{-1}) \exp[\vec{W}^+ A^{-1} \vec{W}], \end{aligned} \quad (38)$$

where $\vec{Z} = (z_1^R + iz_1^I, \dots, z_N^R + iz_N^I)^t$, \vec{W} , \vec{W}^t are two arbitrary complex vectors, A_3 is a matrix with positive Hermitian part.

It is easy to check that

$$\det(A_1) = \det(A_2) = [c^2 - 2(1+m)(2+n)][c'^2 - 2(1+m)(2+n)]. \quad (39)$$

Using Eqs.(38,39) into Eq.(34), hence Eq.(34) requires that

$$\vec{B}^t A_1^{-1} \vec{B} - \vec{B}^t A_2^{-1} \vec{B} = 0. \quad (40)$$

With direct computation, Eq.(40) reads

$$\frac{8ic'}{c^2 - 2(1+m)(2+n)} u^I v^R - \frac{8ic}{c^2 - 2(1+m)(2+n)} u^R v^I = 0. \quad (41)$$

Eq.(41) holds for arbitrary real numbers u^I, v^R, u^R, v^I , thus

$$c = c' = 0. \quad (42)$$

On the other hand, from Eqs.(15,20-22), it is easy to see that

$$\rho^{AB} = \rho^A \otimes \rho^B \Leftrightarrow \chi(\rho^{AB}, \lambda_1, \lambda_2) = \chi(\rho^A, \lambda_1) \chi(\rho^B, \lambda_2). \quad (43)$$

Together with Eqs.(18,19), we see that

$$c = c' = 0 \Leftrightarrow \rho^{AB} = \rho^A \otimes \rho^B. \quad (44)$$

We then complete this proof.

4. Summary

Using the $\text{su}(n)$ algebra, we provided a necessary and sufficient condition for a finite-dimensional bipartite state to be lazy, this condition can be explicitly checked for a given state in terms of the structure constants $\{f_{ijk}\}$ of the $\text{su}(n)$ algebra. We also proved that a two-mode Gaussian states is lazy if and only if it is a direct product state.

How to understand and how to characterize quantum correlation are important questions in quantum information science. Lazy states possess different correlation than entanglement and discord, and have an important dynamics character, i.e., preserving the entropy of subsystem. So the results in this paper are hopefully interesting for the understandings of quantum correlation and designing control schemes of quantum systems.

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References

- [1] R. Horodecki et al., Rev. Mod. Phys. **81**, 865 (2009), and references therein.
- [2] K. Modi, A. Brodutch, H. Cable, T. Paterek, V. Vedral, Rev. Mod. Phys. **84** 1655–1707 (2012), and references therein.
- [3] D. P. Chi, J. S. Kim, and K. Lee, Phys. Rev. A **87** (2013) 062339.
- [4] S.-Y. Liu, Y.-R. Zhang, L.-M. Zhao, W.-L. Yang, and H. Fan, arXiv:1307.4848.
- [5] D. Spehner and M. Orszag, J. Phys. A **47** (2013) 035302.
- [6] B. Cakmak and Z. Gedik, J. Phys. A **46** (2013) 465302.
- [7] C. A. Rodriguez-Rosario, G. Kimura, H. Imai, and A. Aspuru-Guzik, Phys. Rev. Lett. **106** 050403 (2011).
- [8] A. Hutter and S. Wehner, Phys. Rev. Lett. **108** 070501 (2012)
- [9] A. Ferraro, L. Aolita, D. Cavalcanti, F. Cucchietti, and A. Acin, Phys. Rev. A **81** 052318 (2010).
- [10] J. Xu, arXiv:1401.4260 (2014).
- [11] J. Schlienz, and G. Mahler, Phys. Rev. A **52** 4396 (1995).
- [12] G. Mahler and V. A. Weberruβ, Quantum Networks (Springer-Verlag, Berlin, 1998).
- [13] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. **84**, 621 (2012).
- [14] S. Olivares, Eur. Phys. J. Special Topics **203**, 3-24 (2012)
- [15] X.-B. Wang, T. Hiroshima, A. Tomita, and M. Hayashi, Phys. Rep. **448** 1 (2007).
- [16] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84** 2722 (2000).
- [17] R. Simon, Phys. Rev. Lett. **84** 2726 (2000).
- [18] P. Giorda and M. G. A. Paris, Phys. Rev. Lett. **105** 020503 (2010).
- [19] G. Adesso and A. Datta, Phys. Rev. Lett. **105** 030501 (2010).
- [20] R. Simon, N. Mukunda and B. Dutta, Phys. Rev. A **49** 1567 (1994).
- [21] K. E. Cahill and R. J. Glauber, Phys. Rev. **177** 1857 (1969).
- [22] R. J. Glauber, Phys. Rev. **131** 2766 (1963); J. R. Klauder, J. Math. Phys., **4** (1963). 1055.
- [23] E. C. G. Sudarshan, Phys. Rev. Lett. **10** 277 (1963).
- [24] C. L. Mehta, Phys. Rev. Lett. **18** 752 (1967).
- [25] A. Altland and B. Simons, *Condensed Matter Field Theory* (Cambridge University Press) (2006), page 103.